

Exact canonically conjugate momenta approach to a one-dimensional neutron-proton system, I

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Abstract

Introducing collective variables, a collective description of nuclear surface oscillations has been developed with the first quantized language, contrary to the second quantized one in Sunakawa's approach for a Bose system. It overcomes difficulties remaining in the traditional theories of nuclear collective motions: Collective momenta are not exact canonically conjugate to collective coordinates and are not independent. On the contrary to such a description, Tomonaga first gave the basic idea to approach elementary excitations in a one-dimensional Fermi system. The Sunakawa's approach for a Fermi system is also expected to work well for such a problem. In this paper, on the *isospin* space, we define a density operator and further following Tomonaga, introduce a collective momentum. We propose an *exact* canonically momenta approach to a one-dimensional neutron-proton (N-P) system under the use of the Grassmann variables.

Keywords: Collective motion of a one-dimensional neutron-proton system;
exact canonically conjugate momenta; Grassmann variables

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1 Introduction

To study quarks and $SU(N)$ colored non-Abelian gluons (QCD) in the large- N limit, Jevicki and Sakita developed a collective field formalism involving only gauge-invariant variables [1, 2, 3, 4]. Though the QCD describes successfully short-distance phenomena due to the property of asymptotic freedom, one still has the difficult problem of color confinement occurring as a large-distance phenomenon. Their basic idea consists in reformulating the quantum collective field theory in terms of gauge-invariant variables. It leads to an effective Hamiltonian which determines the behavior in the large- N limit by classical stationary point solutions.

On the other hand, in the studies of collective motions in nuclei, the very difficult problems of large-amplitude collective motions, which are strongly nonlinear phenomena in quantum nuclear dynamics, still remain unsolved. How to go beyond the usual mean field theories towards a construction of a theory for large-amplitude collective motions in nuclei [5, 6]?

Applying Tomonaga's idea for collective motion theory [7, 8] to nuclei with the aid of the Sunakawa's discrete integral equation method [9], we developed a collective description of surface oscillations of nuclei [10, 11]. It gives a possible microscopic foundation of nuclear collective motions related to the Bohr-Mottelson model [12]. Introducing collective variables, a collective description is provided by using the first quantized language, contrary to the second quantized one in the Sunakawa's approach for a Bose system. It overcomes the difficulties still remaining in the traditional theoretical treatments of nuclear collective motions: Collective momenta in the Tomonaga's approach are not exact canonically conjugate to collective coordinates and are not independent. Our *exact* canonically conjugate momenta to collective coordinates are found from a viewpoint different from the canonical transformation theory and the group theory [13, 14, 15]. Recently we got *exact* canonical variables, revisiting the Tomonaga's work and described a collective motion also in two-dimensional nuclei [16].

In constructing a collective field theory for an $SU(N)$ quantum system [17, 18], we are standing on a situation similar to the above one. It is regarded as a common feature of strongly nonlinear physics. Applying Tomonaga's idea, a collective description of the $SU(N)$ system is plausible in terms of collective variables invariant under an $SU(N)$ transformation. One of the present authors (S.N.) gave the *exact* canonical variables using the discrete integral equation method [19], which is regarded as a natural extension of the Sunakawa's variables to the variables in the $SU(N)$ system. But they are derived in the first quantized language.

On the contrary to such collective descriptions, to approach elementary excitations in a Fermion system, 65 years ago, Tomonaga first gave another idea [20, 21]. A similar idea was also given by Luttinger with a slight modification of Tomonaga's proposal [22]. Their ideas have the advantage of being exactly solvable [23, 24]. While the Sunakawa's approach for a Fermi system [9] also may be anticipated to work well for such a problem. In this paper, on the *isospin* space (T, T_z) , we define a density operator ρ_k^{T, T_z} and further following Tomonaga, introduce a collective momentum. Then we propose an *exact* canonical momenta approach to a one-dimensional neutron-proton system under the use of the Grassmann variables [25, 26, 27].

In Sec. 2 introducing collective variables $\rho_k^{0,0}$ and their associated variables $\pi_k^{0,0}$, we give commutation relations among them. In Sec. 3, we define *exact* canonically conjugate momenta $\Pi_k^{0,0}$ by a discrete integral equation and devote ourselves to the proof of the *exact*

canonical commutation relation among collective variables $\rho_k^{0,0}$ and $\Pi_k^{0,0}$. In Sec. 4, the dependence of the original Hamiltonian on $\Pi_k^{0,0}$ and $\rho_k^{0,0}$ is determined. Section 5 is devoted to a calculation of a constant term in the collective Hamiltonian. Finally in Sec. 6 some discussions and further perspectives are given. In the Appendix the calculation of some commutators are presented.

2 Collective Variables and the Associated Relations

Let H be the Hamiltonian of a one dimensional Fermion system:

$$H = T + V = \int dx \psi^\dagger(x) \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) \psi(x) + \frac{1}{2} \iint dx dx' \psi^\dagger(x) \psi^\dagger(x') V(x-x') \psi(x') \psi(x), \quad (2.1)$$

where the field operators $\psi(x)$ and $\psi^\dagger(x)$ satisfy the canonical anti-commutation relations,

$$\{\psi(x), \psi^\dagger(x')\} = \delta(x-x'), \quad \{\psi(x), \psi^\dagger(x')\} = 0, \quad \{\psi^\dagger(x), \psi^\dagger(x')\} = 0. \quad (2.2)$$

To deal with a proton-neutron system, let us introduce the z -component of *isospin* to distinguish the Neutron and the Proton:

$$\left. \begin{aligned} \tau_z &= \frac{1}{2}, \quad \text{for Neutron,} \\ \tau_z &= -\frac{1}{2}, \quad \text{for Proton.} \end{aligned} \right\} \quad (2.3)$$

The operators $\psi(x)$ and $\psi^\dagger(x)$ in (2.1) are expanded and separated into two parts according to (2.3), respectively, as

$$\left. \begin{aligned} \psi(x) &= \frac{1}{\sqrt{L}} \sum_{k\tau_z} a_{k\tau_z} e^{ikx} \phi_{\tau_z} = \frac{1}{\sqrt{L}} \sum_{k\tau_z} \frac{1+2\tau_z}{2} a_{k\tau_z} e^{ikx} \phi_{\tau_z} + \frac{1}{\sqrt{L}} \sum_{k\tau_z} \frac{1-2\tau_z}{2} a_{k\tau_z} e^{ikx} \phi_{\tau_z} \\ \psi^\dagger(x) &= \frac{1}{\sqrt{L}} \sum_{k\tau_z} a_{k\tau_z}^\dagger e^{-ikx} \phi_{\tau_z}^* = \frac{1}{\sqrt{L}} \sum_{k\tau_z} \frac{1+2\tau_z}{2} a_{k\tau_z}^\dagger e^{-ikx} \phi_{\tau_z}^* + \frac{1}{\sqrt{L}} \sum_{k\tau_z} \frac{1-2\tau_z}{2} a_{k\tau_z}^\dagger e^{-ikx} \phi_{\tau_z}^* \end{aligned} \right\} \quad (2.4)$$

and the interaction potential $V(x)$ is also expanded as

$$V(x) = \frac{1}{L} \sum_k \nu(k) e^{ikx}. \quad (2.5)$$

Here we have used the following orthogonal relations:

$$\int (e^{ik'x})^* e^{ikx} dx = \int e^{i(k-k')x} dx = L \delta_{k,k'}, \quad \int \phi_{\tau_z'}^* \phi_{\tau_z} d\tau = \delta_{\tau_z', \tau_z}, \quad (2.6)$$

where L is the length of a one-dimensional periodic box. The anti-commutation relations among $a_{k\tau_z}$'s and $a_{k\tau_z}^\dagger$'s are given as

$$\left. \begin{aligned} \{a_{k\tau_z}, a_{k'\tau_z'}^\dagger\} &= \delta_{k,k'} \delta_{\tau_z', \tau_z}, \\ \{a_{k\tau_z}, a_{k'\tau_z'}\} &= \{a_{k\tau_z}^\dagger, a_{k'\tau_z'}^\dagger\} = 0. \end{aligned} \right\} \quad (2.7)$$

To study *isospin* T collective excitations, with the use of the Clebsch-Gordan coefficients $\langle \frac{1}{2} \tau_z \frac{1}{2} \tau_z' | T T_z \rangle$ on the *isospin* space (T, T_z) , we define the Fourier component of the density operator ($\rho(x) = \psi^\dagger(x) \psi(x)$) dependent on the *isospin* space (T, T_z) as

$$\left. \begin{aligned} \rho_k^{T, T_z} &\equiv \frac{\sqrt{2}}{\sqrt{A}} \sum_{p, \tau_z, \tau_z'} \left\langle \frac{1}{2} \tau_z \frac{1}{2} \tau_z' | T T_z \right\rangle a_{p+\frac{k}{2}, \tau_z}^\dagger (-1)^{\frac{1}{2}+\tau_z'} a_{p-\frac{k}{2}, -\tau_z'}, \quad \rho_k^{T, T_z \dagger} = (-1)^{T_z} \rho_{-k}^{T, -T_z}, \\ \rho_0^{0,0} &= \frac{\sqrt{2}}{\sqrt{2A}} \sum_{p, \tau_z} a_{p, \tau_z}^\dagger a_{p, \tau_z} = \frac{1}{\sqrt{A}} (N+Z) = \sqrt{A}, \\ \rho_0^{1,0} &= \frac{\sqrt{2}}{\sqrt{2A}} \sum_p (a_{p,N}^\dagger a_{p,N} - a_{p,P}^\dagger a_{p,P}) = \frac{1}{\sqrt{A}} (N-Z), \\ \rho_0^{1,1} &= -\frac{\sqrt{2}}{\sqrt{A}} \sum_p a_{p,N}^\dagger a_{p,P}, \quad \rho_0^{1,-1} = \frac{\sqrt{2}}{\sqrt{A}} \sum_p a_{p,P}^\dagger a_{p,N}, \end{aligned} \right\} \quad (2.8)$$

where N , Z and A are the total numbers of the neutron, the proton and the N-P System under consideration, respectively [28, 29, 30]. Substituting (2.4) and (2.5) into the expression for the interaction V in (2.1), the V is expressed as

$$V = \frac{1}{2L} \sum_{\{k\}, \{\tau_z\}, T, T_z} \nu_T(k) \langle \frac{1}{2} \tau'_z \frac{1}{2} \tau''_z | TT_z \rangle \langle \frac{1}{2} \tau'''_z \frac{1}{2} \tau''''_z | TT_z \rangle a_{k'\tau'_z}^\dagger a_{k''\tau''_z}^\dagger a_{k''+k, \tau'''_z} a_{k'-k, \tau''''_z} \quad (2.9)$$

$$= \frac{A}{4L} \sum_{T, T_z, k} \left\{ \sum_{T'} (2T'+1) W\left(\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2}; TT'\right) \nu_{T'}(k) \right\} \rho_k^{T, T_z} (-1)^{T_z} \rho_{-k}^{T, -T_z} - \frac{A}{4L} \sum_{T, k} (2T+1) \nu_T(k).$$

Here we have used the relations (2.7) and (2.8). Then the final expression for the Hamiltonian H is given as

$$H = T + V = \sqrt{2} \sum_{k, \tau_z} \frac{\hbar^2 k^2}{2m} (-1)^{\frac{1}{2} - \tau_z} \left\langle \frac{1}{2} \tau_z \frac{1}{2} - \tau_z | 00 \right\rangle a_{k\tau_z}^\dagger a_{k\tau_z} + \frac{A}{4L} \sum_{T, T_z, k} \nu_T^F(k) \rho_k^{T, T_z} (-1)^{T_z} \rho_{-k}^{T, -T_z} - \frac{A}{4L} \sum_{T, k} (2T+1) \nu_T(k), \quad (2.10)$$

$$\nu_T^F(k) \equiv \sum_{T'} (2T'+1) W\left(\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2}; TT'\right) \nu_{T'}(k).$$

Using (2.7), the commutation relation between $\rho_{k_1}^{T_1, T_{z1}}$ and $\rho_{k_2}^{T_2, T_{z2}}$ is calculated as

$$\left[\rho_{k_1}^{T_1, T_{z1}}, \rho_{k_2}^{T_2, T_{z2}} \right] = \frac{\sqrt{2}}{\sqrt{A}} \sum_{T_3, T_{z3}} \{ (-1)^{T_1+T_2+T_3} - 1 \} \times \sqrt{(2T_1+1)(2T_2+1)} W\left(\frac{1}{2} \frac{1}{2} T_1 T_2; T_3 \frac{1}{2}\right) \langle T_1 T_{z1} T_2 T_{z2} | T_3 T_{z3} \rangle \rho_{k_1+k_2}^{T_3, T_{z3}}, \quad (2.11)$$

detailed calculations of which are given in Appendix A. With the aid of (2.10) and (2.11), the commutation relation between V and ρ_k^{T, T_z} is computed as

$$\left[V, \rho_k^{T, T_z} \right] = \frac{\sqrt{A}}{2\sqrt{2}L} \sum_{T_1, T_{z1}, k_1} \nu_{T_1}^F(k_1) \sum_{T_2, T_{z2}} \{ (-1)^{T_1+T_2+T_3} - 1 \} \sqrt{(2T_1+1)(2T_2+1)} \times W\left(\frac{1}{2} \frac{1}{2} T_1 T_2; T \frac{1}{2}\right) \langle T_1 T_{z1} T_2 T_{z2} | TT_z \rangle \left\{ \rho_{k_1}^{T_1, T_{z1}} \rho_{k-k_1}^{T_2, T_{z2}} + \rho_{k+k_1}^{T_2, T_{z2}} \rho_{-k_1}^{T_1, T_{z1}} \right\}. \quad (2.12)$$

In the definition of the Fourier component of the density operator (2.8), we concentrate on the two following cases

(i) : $T=0, T_z=0,$

(ii) : $T=1, T_z=0.$

Then, we have

$$\left. \begin{aligned} \text{(i) : } \rho_k^{0,0} &\equiv \frac{\sqrt{2}}{\sqrt{A}} \sum_{p, \tau_z} \left\langle \frac{1}{2} \tau_z \frac{1}{2} - \tau_z | 00 \right\rangle a_{p+\frac{k}{2}, \tau_z}^\dagger (-1)^{\frac{1}{2} - \tau_z} a_{p-\frac{k}{2}, \tau_z} = \frac{1}{\sqrt{A}} \sum_{p, \tau_z} a_{p+\frac{k}{2}, \tau_z}^\dagger a_{p-\frac{k}{2}, \tau_z}, \\ \rho_k^{0,0\dagger} &= \rho_{-k}^{0,0}, \quad [\rho_k^{0,0}, \rho_{k'}^{0,0}] = 0, \quad [V, \rho_k^{0,0}] = 0, \\ \text{(ii) : } \rho_k^{1,0} &\equiv \frac{\sqrt{2}}{\sqrt{A}} \sum_{p, \tau_z} \left\langle \frac{1}{2} \tau_z \frac{1}{2} - \tau_z | 10 \right\rangle a_{p+\frac{k}{2}, \tau_z}^\dagger (-1)^{\frac{1}{2} - \tau_z} a_{p-\frac{k}{2}, \tau_z} = \frac{2}{\sqrt{A}} \sum_{p, \tau_z} \tau_z a_{p+\frac{k}{2}, \tau_z}^\dagger a_{p-\frac{k}{2}, \tau_z}, \\ \rho_k^{1,0\dagger} &= \rho_{-k}^{1,0}, \quad [\rho_k^{1,0}, \rho_{k'}^{1,0}] = 0, \quad [V, \rho_k^{1,0}] = 0. \end{aligned} \right\} \quad (2.13)$$

It is possible to prove the last relation in the case (ii) of (2.13). Using (2.12), the commutator $[V, \rho_k^{1,0}]$ is calculated as

$$[V, \rho_k^{1,0}] = -\frac{\sqrt{A}}{\sqrt{2}L} \sum_{k_1} \nu_{T_1}^F(k_1) \{ \rho_{k_1}^{1,1} \rho_{k-k_1}^{1,-1} + \rho_{k+k_1}^{1,-1} \rho_{-k_1}^{1,1} - \rho_{k_1}^{1,-1} \rho_{k-k_1}^{1,1} - \rho_{k+k_1}^{1,1} \rho_{-k_1}^{1,-1} \}, \quad (2.14)$$

which vanishes if the condition $\nu_{T_1}^F(k_1) = \nu_{T_1}^F(k+k_1)$ is satisfied. This means $\nu_{T_1}^F(k_1)$ is constant. Then the variables $\rho_k^{0,0}$ and $\rho_k^{1,0}$ become good collective variables. From now on we will find *exact* canonically conjugate momenta to these collective variables. Following Tomonaga, first we introduce collective momenta associated with the $\rho_{-k}^{0,0}$ through

$$\pi_k^{0,0} = \frac{m}{k^2} (\dot{\rho}_{-k}^{0,0}) = \frac{m}{k^2} \frac{i}{\hbar} [H, \rho_{-k}^{0,0}] = \frac{m}{k^2} \frac{i}{\hbar} [T, \rho_{-k}^{0,0}] = \pi_{-k}^{0,0\dagger}, \quad (k \neq 0). \quad (2.15)$$

Calculating the commutator (2.15), we obtain explicit expressions for the associated collective variables $\pi_k^{0,0}$ as

$$\pi_k^{0,0} = -\frac{i\sqrt{2}\hbar}{\sqrt{A}k^2} \sum_{p, \tau_z} p k \langle \frac{1}{2} \tau_z \frac{1}{2} - \tau_z | 00 \rangle a_{p-\frac{k}{2}, \tau_z}^\dagger (-1)^{\frac{1}{2}-\tau_z} a_{p+\frac{k}{2}, \tau_z} = -\frac{i\hbar}{\sqrt{A}k} \sum_{p, \tau_z} p a_{p-\frac{k}{2}, \tau_z}^\dagger a_{p+\frac{k}{2}, \tau_z}, \quad (2.16)$$

where we have used the explicit expression for the kinetic operator T in (2.10) and

$$[T, \rho_{-k}^{0,0}] = \frac{\sqrt{2}}{\sqrt{A}} \frac{\hbar^2}{2m} \sum_{p, \tau_z} \left\{ \left(p - \frac{k}{2} \right)^2 - \left(p + \frac{k}{2} \right)^2 \right\} \langle \frac{1}{2} \tau_z \frac{1}{2} - \tau_z | 00 \rangle a_{p-\frac{k}{2}, \tau_z}^\dagger (-1)^{\frac{1}{2}-\tau_z} a_{p+\frac{k}{2}, \tau_z}. \quad (2.17)$$

At first, this $\pi_k^{0,0}$ is regarded as the collective momentum conjugate to the density operator $\rho_k^{0,0}$ in the sense of Tomonaga [7, 8]. Unfortunately, however, the commutation relations among the variables $\rho_k^{0,0}$ and $\pi_k^{0,0}$, lead to a following result:

$$\left. \begin{aligned} [\rho_k^{0,0}, \rho_{k'}^{0,0}] &= 0, \\ [\pi_k^{0,0}, \rho_{k'}^{0,0}] &= -\frac{i\hbar}{\sqrt{A}} \frac{k'}{k} \rho_{k'-k}^{0,0}, \\ [\pi_k^{0,0}, \pi_{k'}^{0,0}] &= -\frac{i\hbar}{\sqrt{A}kk'} (k^2 - k'^2) \pi_{k+k'}^{0,0}. \end{aligned} \right\} \quad (2.18)$$

These commutation relations have quite the same structures as those of the commutation relations obtained at the first stage in the Sunakawa's discrete integral equation method [9]. As is shown from (2.18), the right-hand side (RHS) of the second line does not take the value $-i\hbar\delta_{kk'}$ and the third one does not vanish. Detailed calculations for them are given in Appendix A. Then from these facts, it is self-evident that the variables $\rho_{k'}^{0,0}$ and $\pi_k^{0,0}$ are not canonically conjugate to each other if we take into account contributions of the order of $\frac{1}{\sqrt{A}}$.

For $T=1$ and $T_z=0$, we have the following commutation relations:

$$\left. \begin{aligned} [\rho_k^{1,0}, \rho_{k'}^{1,0}] &= 0, \quad [\rho_k^{1,0}, \rho_{k'}^{0,0}] = 0, \\ [\pi_k^{1,0}, \rho_{k'}^{1,0}] &= -\frac{i\hbar}{\sqrt{A}} \frac{k'}{k} \rho_{k'-k}^{0,0}, \quad [\pi_k^{1,0}, \rho_{k'}^{0,0}] = -\frac{i\hbar}{\sqrt{A}} \frac{k'}{k} \rho_{k'-k}^{1,0}, \\ [\pi_k^{1,0}, \pi_{k'}^{1,0}] &= -\frac{i\hbar}{\sqrt{A}kk'} (k^2 - k'^2) \pi_{k+k'}^{0,0}, \quad [\pi_k^{1,0}, \pi_{k'}^{0,0}] = -\frac{i\hbar}{\sqrt{A}kk'} (k^2 - k'^2) \pi_{k+k'}^{1,0}. \end{aligned} \right\} \quad (2.19)$$

As is clear the structures of the commutators (2.19), they are shown to have the twisted property in the *isospin* space (T, T_z) , comparing with those of the commutators (2.18). This is a quite different behavior from the behavior for an *isospin*-less Fermion system.

3 Exact Canonically Conjugate Momenta

In order to overcome the difficulties mentioned in the preceding section, we define the *exact* canonically conjugate momenta $\Pi_k^{0,0}$ by

$$\Pi_k^{0,0} = \pi_k^{0,0} - \frac{1}{\sqrt{Ak}} \sum_{p \neq k} p \rho_{p-k}^{0,0} \Pi_p^{0,0} \quad (k \neq 0), \quad \Pi_k^{0,0\dagger} = \Pi_k^{0,0}. \quad (3.1)$$

This type of the discrete integral equation was first presented in the Sunakawa's second quantized collective formalism for an interacting Bose system [9]. It was also proposed by us and one of the present author's (S.N.) in the first quantized manner for a description of a quadrupole type nuclear collective motion [10]. As is clear from the structure of (3.1), the variables $\Pi_k^{0,0}$ are no longer one-body operators but essentially many-body operators. From (3.1), we get the *exact* canonical commutation relations

$$[\rho_k^{0,0}, \rho_{k'}^{0,0}] = 0, \quad [\Pi_k^{0,0}, \rho_{k'}^{0,0}] = -i\hbar \delta_{kk'}, \quad [\Pi_k^{0,0}, \Pi_{k'}^{0,0}] = 0, \quad (3.2)$$

and the commutators among the new $\Pi_k^{0,0}$ and the old $\pi_k^{0,0}$ as

$$[\pi_k^{0,0}, \Pi_{k'}^{0,0}] = \frac{i\hbar}{\sqrt{Ak}} (k + k') \Pi_{k+k'}^{0,0}, \quad (3.3)$$

derivation of (3.3) is given in Appendix B. Following Sunakawa's method, the above *exact* canonical commutation relations are proved also in Appendix B. Thus, we have proved the *exact* canonical commutation relations for $\rho_k^{0,0}$ and $\Pi_k^{0,0}$. The hermiticity property $\Pi_k^{0,0\dagger} = \Pi_{-k}^{0,0}$ can be proved with the help of (2.13), (2.15) and (B. 2).

For $T=1$ and $T_z=0$, we have the following commutation relations:

$$\Pi_k^{1,0} = \pi_k^{1,0} - \frac{1}{\sqrt{Ak}} \sum_{p \neq k} p \rho_{p-k}^{0,0} \Pi_p^{1,0} \quad (k \neq 0), \quad \Pi_k^{1,0\dagger} = \Pi_{-k}^{1,0}, \quad (3.4)$$

$$[\rho_k^{1,0}, \rho_{k'}^{1,0}] = 0, \quad [\Pi_k^{1,0}, \rho_{k'}^{1,0}] = -i\hbar \delta_{kk'}. \quad (3.5)$$

As shown before, the structures of the commutators among $\rho_k^{0,0}, \rho_k^{1,0}, \pi_k^{0,0}$ and $\rho_k^{1,0}$ in (2.19) have the twisted property in the *isospin* space (T, T_z) . Due to this twisted property, unfortunately, the commutators $[\Pi_k^{1,0}, \Pi_{k'}^{1,0}]$ do not vanish. Then, strictly speaking, the $\rho_k^{1,0}$ and $\Pi_k^{1,0}$ are not the *exact* canonical variables with each other. Hereafter we concentrate on the first case (i) : $T=0$ and $T_z=0$ given in (2.13).

4 The Π_k - and ρ_k -Dependence of the Hamiltonian

From the original Hamiltonian (2.1) we derive here a collective Hamiltonian in terms of the *exact* canonical variables $\rho_k^{0,0}$ and $\Pi_k^{0,0}$. The potential part V is already written in terms of $\rho_k^{0,0}$. Our task is therefore to express the kinetic part T in terms of the *exact* canonical variables $\rho_k^{0,0}$ and $\Pi_k^{0,0}$. For this purpose, following Sunakawa's method, first we expand it in a power series of the *exact* canonically conjugate momenta $\Pi_k^{0,0}$ as follows:

$$T = T_0(\rho) + \sum_{p \neq 0} T_1(\rho; p) \Pi_p^{0,0} + \sum_{p \neq 0, q \neq 0} T_2(\rho; p, q) \Pi_p^{0,0} \Pi_q^{0,0} + \cdots, \quad (4.1)$$

where

$$T_2(\rho; p, q) = T_2(\rho; q, p). \quad (4.2)$$

In Eq. (4.1), $T_n (n \neq 0)$ are the unknown expansion coefficients. In order to get their explicit expressions, we take the commutators between T and $\rho_k^{0,0}$. On the other hand, from (2.13) and (2.15), we can calculate directly values of the commutators between T and $\rho_k^{0,0}$ by using the definition (3.1) as follows:

$$\begin{aligned} [T, \rho_{-k}^{0,0}] &= \frac{\hbar}{i} (\dot{\rho}_{-k}^{0,0}) = -\frac{i\hbar k^2}{m} \pi_{-k}^{0,0} \\ &= -\frac{i\hbar k^2}{m} \Pi_{-k}^{0,0} + \frac{i\hbar k}{m\sqrt{A}} \sum_{p \neq -k} p \rho_{p+k}^{0,0} \Pi_p^{0,0}. \end{aligned} \quad (4.3)$$

Using Eq. (B. 2) successively, we can easily obtain the commutators

$$[[T, \rho_k^{0,0}], \rho_{k'}^{0,0}] = \begin{cases} -\frac{\hbar^2 k^2}{m}, & \text{for } k' = -k, \\ \frac{\hbar^2 k k'}{m\sqrt{A}} \rho_{k+k'}^{0,0}, & \text{for } k' \neq -k, \end{cases} \quad (4.4)$$

$$[[[T, \rho_k^{0,0}], \rho_{k'}^{0,0}], \rho_{k''}^{0,0}] = 0, \quad (4.5)$$

and so on. Comparing the above results with the commutators between T of (4.1) and $\rho_k^{0,0}$, we can determine the coefficients $T_n (n \neq 0)$. Then we can express the kinetic part T in terms of the *exact* canonical variables $\rho_k^{0,0}$ and $\Pi_k^{0,0}$ as follows:

$$T = T_0(\rho) + \frac{1}{2m} \sum_k k^2 \Pi_k^{0,0} \Pi_{-k}^{0,0} - \frac{1}{2m\sqrt{A}} \sum_{p \neq 0, q \neq 0, p+q \neq 0} p q \rho_{p+q}^{0,0} \Pi_p^{0,0} \Pi_q^{0,0}. \quad (4.6)$$

Here, we should stress that up to the present stage, all the expressions are derived without any approximation.

Our remaining task in this section is to determine the term $T_0(\rho)$ in (4.1) which depends only on $\rho_k^{0,0}$. For this purpose, we also expand it in a power series of the collective coordinates $\rho_k^{0,0}$ in the form

$$T_0(\rho) = C_0 + \sum_{p \neq 0} C_1(p) \rho_p^{0,0} + \sum_{p \neq 0, q \neq 0} C_2(p, q) \rho_p^{0,0} \rho_q^{0,0} + \cdots, \quad (4.7)$$

where $C_2(p, q) = C_2(q, p)$. The expansion coefficients should be determined by a procedure similar to the one used in the previous section. From the definition (3.1), we get easily the discrete integral equation

$$[\Pi_k^{0,0}, T_0(\rho)] = f_k(\rho) - \frac{1}{\sqrt{A}k} \sum_{p \neq 0, k} p \rho_{p-k}^{0,0} [\Pi_p^{0,0}, T_0(\rho)], \quad (4.8)$$

and the inhomogeneous term $f_k(\rho)$ becomes

$$f_k(\rho) \equiv [\pi_k^{0,0}, T_0(\rho)] - \left[\pi_k^{0,0}, T - \frac{1}{2m} \sum_p p^2 \Pi_p^{0,0} \Pi_{-p}^{0,0} + \frac{1}{2m\sqrt{A}} \sum_{p \neq 0, q \neq 0, p+q \neq 0} pq \rho_{p+q}^{0,0} \Pi_p^{0,0} \Pi_q^{0,0} \right] \quad (4.9)$$

$$= [\pi_k^{0,0}, T] - \frac{i\hbar}{mA} \sum_{p \neq 0, q \neq 0} pq \rho_{p+q-k}^{0,0} \Pi_p^{0,0} \Pi_q^{0,0}, \quad (k \neq 0)$$

with the aid of the result (4.6) and the commutation relation (3.3). At a first glance, the operator-valued function $f_k(\rho)$ seems to be dependent on $\Pi_k^{0,0}$. However, it turns out that $f_k(\rho)$ does not really depend on $\Pi_k^{0,0}$ because it commutes with $\rho_k^{0,0}$ as shown below

$$\begin{aligned} [\rho_k^{0,0}, f_k(\rho)] &= [\rho_k^{0,0}, [\pi_k^{0,0}, T]] + \frac{2\hbar^2 k'}{mA} \sum_{p \neq 0} p \rho_{p-(k-k')}^{0,0} \Pi_p^{0,0} \\ &= -2\hbar^2 \frac{k'(k-k')}{m\sqrt{A}} \pi_{k-k'}^{0,0} + \frac{2\hbar^2 k'}{mA} \sum_{p \neq 0} p \rho_{p-(k-k')}^{0,0} \Pi_p^{0,0} \\ &= -2\hbar^2 \frac{k'(k-k')}{m\sqrt{A}} \left\{ \pi_{k-k'}^{0,0} - \frac{1}{\sqrt{A}(k-k')} \sum_{p \text{ all}} p \rho_{p-(k-k')}^{0,0} \Pi_p^{0,0} \right\} = 0. \end{aligned} \quad (4.10)$$

In the above we have used the commutation relation

$$[\rho_k^{0,0}, [\pi_k^{0,0}, T]] = -2\hbar^2 \frac{k'(k-k')}{m\sqrt{A}} \pi_{k-k'}^{0,0}, \quad (4.11)$$

which is proved with the help of the commutation relation between $\pi_k^{0,0}$ and T ,

$$[\pi_k^{0,0}, T] = -\frac{i\hbar}{\sqrt{A}} \frac{\hbar^2}{2m} \frac{1}{k} \sum_{p, \tau_z} p \left\{ \left(p + \frac{k}{2} \right)^2 - \left(p - \frac{k}{2} \right)^2 \right\} a_{p-\frac{k}{2}, \tau_z}^\dagger a_{p+\frac{k}{2}, \tau_z} = -\frac{i\hbar^3}{m\sqrt{A}} \sum_{p, \tau_z} p^2 a_{p-\frac{k}{2}, \tau_z}^\dagger a_{p+\frac{k}{2}, \tau_z} \quad (4.12)$$

Up to the present stage, all the expressions are exact.

From now on, we make an approximation to calculate $T_0(\rho)$ up to the order of $\frac{1}{A}$. First we use $\rho_0^{0,0} = \sqrt{A}$ and $\Pi_k^{0,0} \cong \pi_k^{0,0}$ and give approximate expressions for $\pi_k^{0,0}$ and $\rho_k^{0,0}$ as

$$\pi_k^{0,0} \cong -\frac{i\hbar}{2} \sum_{\tau_z} \left(\bar{\theta} a_{k, \tau_z} - a_{-k, \tau_z}^\dagger \theta \right), \quad \rho_k^{0,0} \cong \sum_{\tau_z} \left(\bar{\theta} a_{-k, \tau_z} + a_{k, \tau_z}^\dagger \theta \right), \quad (4.13)$$

and we regard the operators a_{0, τ_z} and a_{0, τ_z}^\dagger as c -numbers but with the Grassmann variables

$$a_{0, \tau_z} \cong \sqrt{A} \theta, \quad a_{0, \tau_z}^\dagger \cong \sqrt{A} \bar{\theta}, \quad (4.14)$$

where the θ and $\bar{\theta}$ are Grassmann variables and anti-commute with a_{k, τ_z} and a_{k, τ_z}^\dagger [25, 26, 27]. These Grassmann variables play crucial roles to estimate the inhomogeneous term $f_k(\rho)$ (4.9), though they were unnecessary to compute the Sunakawa's $f_k(\rho)$ in the case of a Bose system [9]. Then the second term in the last line of (4.9) is approximately computed as

$$\begin{aligned} & -\frac{i\hbar}{mA} \sum_{p \neq 0, q \neq 0} p \cdot q \cdot \rho_{p+q-k}^{0,0} \Pi_p^{0,0} \Pi_q^{0,0} \cong -\frac{i\hbar}{m\sqrt{A}} \sum_{p \neq 0, p \neq k} p(k-p) \pi_p^{0,0} \pi_{k-p}^{0,0} \\ &= \frac{i\hbar^3}{4m\sqrt{A}} \sum_{p \neq 0, p \neq k} p(k-p) \sum_{\tau_z} \left(\bar{\theta} a_{p, \tau_z} - a_{-p, \tau_z}^\dagger \theta \right) \sum_{\tau'_z} \left(\bar{\theta} a_{k-p, \tau'_z} - a_{-(k-p), \tau'_z}^\dagger \theta \right) + O\left(\frac{1}{A}\right) \\ &= \frac{i\hbar^3}{4m\sqrt{A}} \sum_{p \neq 0, p \neq k} p(k-p) \left(\rho_{-p}^{0,0} - 2 \sum_{\tau_z} a_{-p, \tau_z}^\dagger \theta \right) \left(\rho_{-(k-p)}^{0,0} - 2 \sum_{\tau'_z} a_{-(k-p), \tau'_z}^\dagger \theta \right) + O\left(\frac{1}{A}\right) \quad (4.15) \\ &\cong \theta \bar{\theta} \frac{i\hbar^3}{m\sqrt{A}} \sum_{p, \tau_z} p^2 a_{p-\frac{k}{2}, \tau_z}^\dagger a_{p+\frac{k}{2}, \tau_z} - \theta \bar{\theta} \frac{i\hbar^3 k^2}{4m} \rho_{-k}^{0,0} + \frac{i\hbar^3}{4m\sqrt{A}} \sum_{p \neq 0, p \neq k} p(k-p) \rho_{-p}^{0,0} \rho_{-(k-p)}^{0,0} \\ &\quad + \theta \bar{\theta} \frac{i\hbar^3}{m\sqrt{A}} \sum_{p, \tau_z \neq \tau'_z} \left(p^2 - \frac{k^2}{4} \right) a_{p-\frac{k}{2}, \tau_z}^\dagger a_{p+\frac{k}{2}, \tau'_z} - \theta \bar{\theta} \frac{i\hbar^3}{m\sqrt{A}} \sum_p p^2 + O\left(\frac{1}{A}\right), \end{aligned}$$

in the last line of the above Eq. (4.15), we have used the relations (4.14) and the explicit expression for $\rho_{-k}^{0,0}$ given by the first equation of (2.13).

Putting the calculated results of (4.12) and (4.15) into the $f_k(\rho)$ (4.9) and discarding the first term in the last line of the Eq. (4.15), we can obtain an approximate expression for the operator-valued function $f_k(\rho)$ up to the order of $\frac{1}{\sqrt{A}}$ in the following form:

$$f_k(\rho) = -\frac{i\hbar^3 k^2}{4m} \rho_{-k}^{0,0} + \frac{i\hbar^3}{4m\sqrt{A}k} \sum_{p \neq 0, p \neq k} p(k^2 - pk) \rho_{-p}^{0,0} \rho_{p-k}^{0,0} - \frac{i\hbar^3}{m\sqrt{A}} \sum_p p^2 + O\left(\frac{1}{A}\right), \quad (4.16)$$

if the Grassmann variables θ and $\bar{\theta}$ satisfy a condition $\theta\bar{\theta} = 1$. This is just the same form as that obtained by Sunakawa [9] except that there exists the last constant term proportional to $\sum_p p^2$. The condition $\theta\bar{\theta} = 1$, however, is not good for a realistic Fermi system because the occupation number of the zero-momentum state $\sum_{p,\tau_z} a_{p,\tau_z}^\dagger a_{p,\tau_z} (p=0)$ is not equal to the total number A of the N-P system under consideration. This is quite a different situation from that in the Bose system treated by Sunakawa. The first idea of regarding the operators $a_{k=0}$ and $a_{k=0}^\dagger$ as a c -number \sqrt{N} (This N is the total number of Bose particles) at extremely low temperature was proposed by Bogoliubov [31]. For the present moment, we can't help using this condition. Then substituting (4.16) into (4.8), we can rewrite the RHS of the discrete integral Eq. (4.8) as

$$[\Pi_k^{0,0}, T_0(\rho)] = -\frac{i\hbar^3 k^2}{4m} \rho_{-k}^{0,0} + \frac{i\hbar^3}{4m\sqrt{A}k} \sum_{p \neq 0, p \neq k} p(k^2 - pk + p^2) \rho_{-p}^{0,0} \rho_{p-k}^{0,0} + O\left(\frac{1}{A\sqrt{A}}\right). \quad (k \neq 0) \quad (4.17)$$

From (4.17) and the commutation relations (2.18) and (3.2), we get

$$\left. \begin{aligned} [\Pi_{k'}^{0,0}, [\Pi_k^{0,0}, T_0(\rho)]] &= -\frac{\hbar^4 k^2}{4m} \delta_{k',-k} + \frac{\hbar^4}{4m\sqrt{A}} (k^2 + kk' + k'^2) \rho_{-k-k'}^{0,0}, \\ [\Pi_{k''}^{0,0}, [\Pi_{k'}^{0,0}, [\Pi_k^{0,0}, T_0(\rho)]]] &= -\frac{i\hbar^5}{4m\sqrt{A}} (k^2 + kk' + k'^2) \delta_{k'',-k-k'}, \\ [\Pi_{k'''}^{0,0}, [\Pi_{k''}^{0,0}, [\Pi_{k'}^{0,0}, [\Pi_k^{0,0}, T_0(\rho)]]]] &= 0. \end{aligned} \right\} \quad (4.18)$$

By a procedure similar to the one in Sec. 4, we determine the coefficients $C_n (n \neq 0)$ and then get an approximate form of $T_0(\rho)$ in terms of variables $\rho_k^{0,0}$ in the following form:

$$\begin{aligned} T_0(\rho) = & C_0 + \frac{\hbar^2}{8m} \sum_{p \neq 0} p^2 \rho_p^{0,0} \rho_{-p}^{0,0} \\ & - \frac{\hbar^2}{24m\sqrt{A}} \sum_{p \neq 0, q \neq 0, p+q \neq 0} (p^2 + pq + q^2) \rho_p^{0,0} \rho_q^{0,0} \rho_{-p-q}^{0,0} + O\left(\frac{1}{A}\right). \end{aligned} \quad (4.19)$$

Using the following identities:

$$\left. \begin{aligned} \sum_{p \neq 0, q \neq 0, p+q \neq 0} p^2 \rho_p^{0,0} \rho_q^{0,0} \rho_{-p-q}^{0,0} &= \sum_{p \neq 0, q \neq 0, p+q \neq 0} (p+q)^2 \rho_p^{0,0} \rho_q^{0,0} \rho_{-p-q}^{0,0}, \\ \sum_{p \neq 0, q \neq 0, p+q \neq 0} p^2 \rho_p^{0,0} \rho_q^{0,0} \rho_{-p-q}^{0,0} &= -2 \sum_{p \neq 0, q \neq 0, p+q \neq 0} pq \rho_p^{0,0} \rho_q^{0,0} \rho_{-p-q}^{0,0}, \end{aligned} \right\} \quad (4.20)$$

the lowest kinetic term of T , $T_0(\rho)$ (4.19) is rewritten as

$$T_0(\rho) = C_0 + \frac{\hbar^2}{8m} \sum_{p \neq 0} p^2 \rho_p^{0,0} \rho_{-p}^{0,0} + \frac{\hbar^2}{8m\sqrt{A}} \sum_{p \neq 0, q \neq 0, p+q \neq 0} pq \rho_p^{0,0} \rho_q^{0,0} \rho_{-p-q}^{0,0}, \quad (4.21)$$

where the constant term C_0 remains undetermined yet. Then the detailed calculation of C_0 will be given in the next section.

5 Calculation of the Constant Term

In this section, we calculate the constant term C_0 , the first term in the RHS of (4.21). Substituting (4.21) into (4.6), the constant term C_0 is computed up to the order of $\frac{1}{A}$:

$$C_0 = T - \frac{\hbar^2}{8m} \sum_k k^2 \rho_k^{0,0} \rho_{-k}^{0,0} - \frac{1}{2m} \sum_k k^2 \Pi_k^{0,0} \Pi_{-k}^{0,0} + \frac{1}{2m\sqrt{A}} \sum_{k \neq 0, p \neq 0, k+p \neq 0} k p \rho_{k+p}^{0,0} \Pi_k^{0,0} \Pi_p^{0,0} - \frac{\hbar^2}{8m\sqrt{A}} \sum_{k \neq 0, p \neq 0, k+p \neq 0} k p \rho_{-k-p}^{0,0} \rho_k^{0,0} \rho_p^{0,0}. \quad (5.1)$$

Using $\Pi_k^{0,0} \cong \pi_k^{0,0}$, $\rho_k^{0,0} \cong \sum_{\tau_z} (\bar{\theta} a_{-k, \tau_z} + a_{k, \tau_z}^\dagger \theta)$ and $\pi_k^{0,0} \cong -\frac{i\hbar}{2} \sum_{\tau_z} (\bar{\theta} a_{k, \tau_z} - a_{-k, \tau_z}^\dagger \theta)$, first we can calculate the third term in (5.1) similarly to the calculation of (4.15) and get a result as

$$\begin{aligned} & -\frac{1}{2m} \sum_k k^2 \Pi_k^{0,0} \Pi_{-k}^{0,0} \cong -\frac{1}{2m} \sum_k k^2 \pi_k^{0,0} \pi_{-k}^{0,0} \\ & = \frac{\hbar^2}{8m} \sum_k k^2 \sum_{\tau_z} (\bar{\theta} a_{k, \tau_z} - a_{-k, \tau_z}^\dagger \theta) \sum_{\tau'_z} (\bar{\theta} a_{-k, \tau'_z} - a_{k, \tau'_z}^\dagger \theta) \\ & = \frac{\hbar^2}{8m} \sum_k k^2 \left(\rho_{-k}^{0,0} - 2 \sum_{\tau_z} a_{-k, \tau_z}^\dagger \theta \right) \left(\rho_k^{0,0} - 2 \sum_{\tau'_z} a_{k, \tau'_z}^\dagger \theta \right) \\ & = \frac{\hbar^2}{8m} \sum_k k^2 \rho_k^{0,0} \rho_{-k}^{0,0} - \theta \bar{\theta} \sum_k \frac{\hbar^2 k^2}{2m} \sum_{\tau_z, \tau'_z} a_{k, \tau_z}^\dagger a_{k, \tau'_z} + \theta \bar{\theta} \frac{\hbar^2}{2m} \sum_k k^2 + O\left(\frac{1}{A}\right). \end{aligned} \quad (5.2)$$

As for the case of the forth and fifth terms in (5.1), contrast to the case of the first term, due to the properties $\theta\theta=0$ and $\bar{\theta}\bar{\theta}=0$ we simply have

$$\begin{aligned} & \frac{1}{2m\sqrt{A}} \sum_{k \neq 0, p \neq 0, k+p \neq 0} k p \rho_{k+p}^{0,0} \pi_k^{0,0} \pi_p^{0,0} \\ & \cong -\frac{\hbar^2}{8m\sqrt{A}} \sum_{k \neq 0, p \neq 0, k+p \neq 0} k p \sum_{\tau_z} (\bar{\theta} a_{-k-p, \tau_z} + a_{k+p, \tau_z}^\dagger \theta) \sum_{\tau'_z} (\bar{\theta} a_{k, \tau'_z} - a_{-k, \tau'_z}^\dagger \theta) \sum_{\tau''_z} (\bar{\theta} a_{p, \tau''_z} - a_{-p, \tau''_z}^\dagger \theta) \end{aligned} \quad (5.3)$$

$=0$,

and

$$\begin{aligned} & -\frac{\hbar^2}{8m\sqrt{A}} \sum_{k \neq 0, p \neq 0, k+p \neq 0} k p \rho_{-k-p}^{0,0} \rho_k^{0,0} \rho_p^{0,0} \\ & \cong -\frac{\hbar^2}{8m\sqrt{A}} \sum_{k \neq 0, p \neq 0, k+p \neq 0} k p \sum_{\tau_z} (\bar{\theta} a_{k+p, \tau_z} + a_{-k-p, \tau_z}^\dagger \theta) \sum_{\tau'_z} (\bar{\theta} a_{-k, \tau'_z} - a_{k, \tau'_z}^\dagger \theta) \sum_{\tau''_z} (\bar{\theta} a_{-p, \tau''_z} - a_{p, \tau''_z}^\dagger \theta) \end{aligned} \quad (5.4)$$

$=0$.

Substituting Eqs. (5.2) ~ (5.4) into (5.1), we have a result

$$\begin{aligned} C_0 & \cong (1 - \theta\bar{\theta})T + \theta\bar{\theta} \frac{\hbar^2}{2m} \sum_k k^2 - \theta\bar{\theta} \sum_{k, \tau_z \neq \tau'_z} \frac{\hbar^2 k^2}{2m} a_{k, \tau_z}^\dagger a_{k, \tau'_z} \\ & \cong \frac{\hbar^2}{2m} \sum_k k^2. \end{aligned} \quad (5.5)$$

Here we have used the relation $\theta\bar{\theta}=1$ and neglected $\sum_{k, \tau_z \neq \tau'_z} \frac{\hbar^2 k^2}{2m} a_{k, \tau_z}^\dagger a_{k, \tau'_z}$ in the first line which does not exist in the case of the *isospin*-less Fermion system. The result (5.5) is not identical with the Sunakawa's result [9] for a Bose system. This is because we have dealt with a Fermi system. Then we get a result which is considered as the natural consequence for a Fermi system. It is surprising to see that the C_0 (5.5) coincides with the constant term in the resultant ground state energy given by the Tomonaga's method [20].

6 Discussions and Further Perspectives

Using (4.6), (4.21) and (5.5) and separating the term $C_0 \cong \sum_k \frac{\hbar^2 k^2}{2m}$ into two parts $-\sum_k \frac{\hbar^2 k^2}{4m}$ and $\frac{3}{2} \sum_k \frac{\hbar^2 k^2}{2m}$ and denoting $\nu_{T=0}^F(k)$ and $\nu_{T=0}(k)$ simply as $\nu^F(k)$ and $\nu(k)$, we can express the original Hamiltonian H in terms of the *exact* canonical variables $\rho_k^{0,0}$ and $\Pi_k^{0,0}$ as

$$H = -\frac{A(A+2)}{8L} \nu(0) - \sum_k \frac{\hbar^2 k^2}{4m} - \frac{A}{4L} \sum_{k \neq 0} \nu(k) + \sum_{k \neq 0} \left\{ \frac{k^2}{2m} \Pi_k^{0,0} \Pi_{-k}^{0,0} + \left(\frac{\hbar^2 k^2}{8m} - \frac{A}{8L} \nu(k) \right) \rho_k^{0,0} \rho_{-k}^{0,0} \right\} - \frac{1}{2m\sqrt{A}} \sum_{p \neq 0, q \neq 0, p+q \neq 0} pq \rho_{p+q}^{0,0} \Pi_p^{0,0} \Pi_q^{0,0} + \frac{\hbar^2}{8m\sqrt{A}} \sum_{p \neq 0, q \neq 0, p+q \neq 0} pq \rho_p^{0,0} \rho_q^{0,0} \rho_{-p-q}^{0,0} + \frac{3}{2} \sum_k \frac{\hbar^2 k^2}{2m}, \quad (6.1)$$

where we have used the relation $\nu^F(k) = -\frac{1}{2} \nu(k)$. This is Sunakawa's form up to the order of $\frac{1}{\sqrt{A}}$ [9], except the last term $\frac{3}{2} \sum_k \frac{\hbar^2 k^2}{2m}$ in the RHS of (6.1). This difference arises due to the fact that we deal with a Fermi system but not a Bose system. At the present moment, we discard this term. The sum of the three terms in the first line and the two terms of the second line in (6.1) are considered as the lowest order Hamiltonian H_0

$$H_0 = -\frac{A(A+2)}{8L} \nu(0) + \sum_{k \neq 0} \left\{ -\frac{\hbar^2 k^2}{4m} - \frac{A}{4L} \nu(k) + \frac{k^2}{2m} \Pi_k^{0,0} \Pi_{-k}^{0,0} + \left(\frac{\hbar^2 k^2}{8m} - \frac{A}{8L} \nu(k) \right) \rho_k^{0,0} \rho_{-k}^{0,0} \right\}. \quad (6.2)$$

Now, let us introduce the Boson annihilation and creation operators defined as

$$\left. \begin{aligned} \alpha_k &\equiv \sqrt{\frac{mE_k}{2\hbar^2 k^2}} \rho_{-k}^{0,0} + \frac{ik}{\sqrt{2mE_k}} \Pi_k^{0,0}, \quad (k \neq 0), \\ \alpha_k^\dagger &\equiv \sqrt{\frac{mE_k}{2\hbar^2 k^2}} \rho_k^{0,0} - \frac{ik}{\sqrt{2mE_k}} \Pi_{-k}^{0,0}, \quad (k \neq 0). \end{aligned} \right\} \quad (6.3)$$

Using (6.3) and (4.13), the *exact* canonical collective variables $\rho_k^{0,0}$ and $\Pi_k^{0,0}$ are expressed as

$$\left. \begin{aligned} \rho_k^{0,0} &= \sqrt{\frac{\hbar^2 k^2}{2mE_k}} \frac{1}{2} (\alpha_{-k} + \alpha_k^\dagger) = \sum_{\tau_z} \bar{\theta} a_{-k, \tau_z} + \sum_{\tau_z} a_{k, \tau_z}^\dagger \theta, \\ \Pi_k^{0,0} &= -i \frac{\sqrt{2mE_k}}{k} \frac{1}{2} (\alpha_k - \alpha_{-k}^\dagger) = -\frac{i\hbar}{2} \left(\sum_{\tau_z} \bar{\theta} a_{k, \tau_z} - \sum_{\tau_z} a_{-k, \tau_z}^\dagger \theta \right), \quad (k \neq 0), \end{aligned} \right\} \quad (6.4)$$

substituting which into (6.2), then the lowest order Hamiltonian H_0 (6.2) is diagonalized as

$$H_0 = -\frac{A(A+2)}{8L} \nu(0) + \sum_{k \neq 0} E_k \alpha_k^\dagger \alpha_k, \quad E_k \equiv \sqrt{(\varepsilon_k)^2 + \frac{\hbar^2 k^2}{m} \frac{A}{2L} \nu(k)}, \quad \varepsilon_k \equiv \frac{\hbar^2 k^2}{2m}. \quad (6.5)$$

From (6.3)-(6.5), we have a Bogoliubov transformation for Boson-like operators $\sum_{\tau_z} \bar{\theta} a_{k, \tau_z}$ and $\sum_{\tau_z} a_{k, \tau_z}^\dagger \theta$ as

$$\left. \begin{aligned} \alpha_k &= \frac{1}{2\sqrt{\varepsilon_k E_k}} \left[(E_k + \varepsilon_k) \sum_{\tau_z} \bar{\theta} a_{k, \tau_z} + (E_k - \varepsilon_k) \sum_{\tau_z} a_{-k, \tau_z}^\dagger \theta \right], \\ \alpha_{-k}^\dagger &= \frac{1}{2\sqrt{\varepsilon_k E_k}} \left[(E_k - \varepsilon_k) \sum_{\tau_z} \bar{\theta} a_{k, \tau_z} + (E_k + \varepsilon_k) \sum_{\tau_z} a_{-k, \tau_z}^\dagger \theta \right], \end{aligned} \right\} \quad (6.6)$$

which is the same as the famous Bogoliubov transformation for the usual Bosons [31, 32]. The diagonalization (6.5) has also been given for the usual Bose system by Sunakawa [33].

To convert the Hamiltonian (6.1) into a coordinate representation, we introduce collective field variables $\hat{\rho}(x)$ and $\hat{\rho}^\dagger(x)$ ($\hat{\rho}(x) = n + \hat{\rho}^\dagger(x)$, $n = \frac{A}{L}$) and $\Pi(x)$ defined by the Fourier transformation of the *exact* canonical variables $\rho_k^{0,0}$ and $\Pi_k^{0,0}$ as

$$\hat{\rho}(x) \equiv \frac{\sqrt{A}}{L} \sum_k \rho_k e^{ikx}, \quad \hat{\rho}^\dagger(x) \equiv \frac{\sqrt{A}}{L} \sum_{k \neq 0} \rho_k e^{ikx}, \quad \Pi(x) \equiv \frac{1}{\sqrt{A}} \sum_{k \neq 0} \Pi_k e^{-ikx}, \quad (6.7)$$

which leads to the following coordinate representation of the collective field Hamiltonian:

$$H = \sum_k \frac{\hbar^2 k^2}{2m} + \int dx V(x) \hat{\rho}(x) + \int dx \left[\frac{m}{2} \partial_x \Pi(x) \cdot \hat{\rho}(x) \cdot \partial_x \Pi(x) + \frac{\hbar^2}{8mn} \left(1 - \frac{\hat{\rho}'(x)}{n} + \frac{\hat{\rho}''(x)}{n^2} \right) \partial_x \hat{\rho}(x) \partial_x \hat{\rho}(x) \right], \quad (6.8)$$

which is equal to Sunakawa's result for a Bose system except a difference of the first term $\sum_k \frac{\hbar^2 k^2}{2m}$ from the corresponding term $-\sum_k \frac{\hbar^2 k^2}{4m}$ given in [34]. In (6.8), we must emphasize that the term $\sum_k \frac{\hbar^2 k^2}{2m}$ is interpreted as a kinetic energy in the state of perfect degeneracy if the term $\int dx \frac{m}{2} \partial_x \Pi(x) \cdot \hat{\rho}(x) \cdot \partial_x \Pi(x)$ is regarded as the "excitation kinetic energy" in the sense of Tomonaga [20]. The collective field Hamiltonian (6.8) also was given by one of the present author's (S.N.) in his *exact* canonically conjugate momenta approach to an $SU(N)$ quantum system [19]. Contrary to such collective descriptions, in the beginning we already referred to another way to study of collective motions: Tomonaga first developed a quite different approach to an elementary excitation in a Fermi system [20]. A similar attempt was also given by Luttinger [22]. To solve the Luttinger's model exactly, Matias and Lieb made a field theoretical approach based on the fact that density operators ρ_k define a Bose field which is *ipso facto* associated with a Fermi-Dirac field. They obtained the exact and nontrivial spectrum [35]. Adding to such a historical achievement, Sunakawa's method for a Fermi system [9] may be anticipated to work well for the above mentioned problems. To carry out such a strategy, following Tomonaga [20], we separate a density operator $\rho_k^{0,0}$ into two parts as $\rho_k^{0,0} = \rho_k^{0,0(+)} + \rho_k^{0,0(-)}$ where $\rho_k^{0,0(+)}$ and $\rho_k^{0,0(-)}$ are defined as

$$\rho_k^{0,0(+)} \equiv \frac{1}{\sqrt{A}} \sum_{p>0, \tau_z} a_{p+\frac{k}{2}, \tau_z}^\dagger a_{p-\frac{k}{2}, \tau_z}, \quad \rho_k^{0,0(-)} \equiv \frac{1}{\sqrt{A}} \sum_{p<0, \tau_z} a_{p+\frac{k}{2}, \tau_z}^\dagger a_{p-\frac{k}{2}, \tau_z}. \quad (6.9)$$

Further according to Tomonaga [20], we introduce collective momenta $\pi_k^{0,0(+)}$ and $\pi_k^{0,0(-)}$ defined as

$$\pi_k^{0,0(\pm)} \equiv \frac{m}{k^2} \left(\dot{\rho}_{-k}^{0,0(\pm)} \right) = \frac{m}{k^2} \frac{i}{\hbar} [H, \rho_{-k}^{0,0(\pm)}] = \frac{m}{k^2} \frac{i}{\hbar} [T, \rho_{-k}^{0,0(\pm)}], \quad (k \neq 0), \quad (6.10)$$

and the momentum operator $\pi_k^{0,0}$ is expressed as $\pi_k^{0,0} = \pi_k^{0,0(+)} + \pi_k^{0,0(-)}$. Calculating the commutator (6.10), we obtain the explicit expressions for the collective variables $\pi_k^{0,0(\pm)}$ as

$$\pi_k^{0,0(+)} = -\frac{i\hbar}{\sqrt{Ak}} \sum_{p>0, \tau_z} p a_{p-\frac{k}{2}, \tau_z}^\dagger a_{p+\frac{k}{2}, \tau_z}, \quad \pi_k^{0,0(-)} = -\frac{i\hbar}{\sqrt{Ak}} \sum_{p<0, \tau_z} p a_{p-\frac{k}{2}, \tau_z}^\dagger a_{p+\frac{k}{2}, \tau_z}. \quad (6.11)$$

Under the above preliminaries for introducing the collective variables, an *exact* canonical momenta approach to the one-dimensional neutron-proton system may be developed. More specifically, the interesting problem of describing the elementary excitations occurring in a rod at *isospin* $T=0$, the dipole oscillations of one-dimensional nuclei, the Goldhaber-Teller model [36] or the Steinwedel-Jensen model [37] may be considered. See also Refs. [38] and [39]. By applying the *exact* canonical momenta approach to these models, a better description of the elementary energy excitations is expected to be obtained, because that approach is designed to take into account essential many-body effects, which were not considered in previous treatments of these models. In this context, a new field of exploration of elementary excitations of a one-dimensional Fermi system, which is intended to be presented elsewhere, may be opened.

Finally, we emphasize the possibility of extending the present *exact* canonical momenta approach to the three-dimensional case which would lead to the *isospin* $T=0$ quantum hydrodynamics, by introducing the velocity operator $\mathbf{v}_{\mathbf{k}}$ instead of canonical conjugate momentum $\mathbf{\Pi}_{\mathbf{k}}$ as has been done by Sunakawa [33, 34].

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Appendix

A Commutators Among $\rho_{k_1}^{T_1, T_{z1}}$ and $\rho_{k_2}^{T_2, T_{z2}}$ and Derivation of (2.18) and (3.3)

Using the first relation of (2.8), commutators among $\rho_{k_1}^{T_1, T_{z1}}$ and $\rho_{k_2}^{T_2, T_{z2}}$ are calculated as

$$\begin{aligned}
& \left[\rho_{k_1}^{T_1, T_{z1}}, \rho_{k_2}^{T_2, T_{z2}} \right] \\
&= \frac{2}{A} \sum_{p_1, \tau_{z1}, \tau'_{z1}} \sum_{p_2, \tau_{z2}, \tau'_{z2}} \left\langle \frac{1}{2} \tau_{z1} \frac{1}{2} \tau'_{z1} | T_1 T_{z1} \right\rangle \left\langle \frac{1}{2} \tau_{z2} \frac{1}{2} \tau'_{z2} | T_2 T_{z2} \right\rangle (-1)^{\frac{1}{2} + \tau'_{z1}} (-1)^{\frac{1}{2} + \tau'_{z2}} \\
&\times \left(a_{p_1 + \frac{k_1}{2}, \tau_{z1}}^\dagger a_{p_2 - \frac{k_2}{2}, -\tau'_{z2}} \delta_{p_1 - \frac{k_1}{2}, p_2 + \frac{k_2}{2}} \delta_{-\tau'_{z1}, \tau_{z2}} - a_{p_2 + \frac{k_2}{2}, \tau_{z2}}^\dagger a_{p_1 - \frac{k_1}{2}, -\tau'_{z1}} \delta_{p_1 + \frac{k_1}{2}, p_2 - \frac{k_2}{2}} \delta_{\tau_{z1}, -\tau'_{z2}} \right) \\
&= \frac{2}{A} \sum_p \sum_{\tau_{z1}, \tau_{z2}, \tau'_{z2}} \left\langle \frac{1}{2} \tau_{z1} \frac{1}{2} -\tau_{z2} | T_1 T_{z1} \right\rangle \left\langle \frac{1}{2} \tau_{z2} \frac{1}{2} \tau'_{z2} | T_2 T_{z2} \right\rangle (-1)^{\frac{1}{2} - \tau_{z2}} (-1)^{\frac{1}{2} + \tau'_{z2}} a_{p + \frac{k_1 + k_2}{2}, \tau_{z1}}^\dagger a_{p - \frac{k_1 + k_2}{2}, -\tau'_{z2}} \\
&- \frac{2}{A} \sum_p \sum_{\tau_{z1}, \tau_{z2}, \tau'_{z2}} \left\langle \frac{1}{2} \tau_{z1} \frac{1}{2} \tau'_{z1} | T_1 T_{z1} \right\rangle \left\langle \frac{1}{2} \tau_{z2} \frac{1}{2} -\tau_{z1} | T_2 T_{z2} \right\rangle (-1)^{\frac{1}{2} + \tau'_{z1}} (-1)^{\frac{1}{2} - \tau_{z1}} a_{p + \frac{k_1 + k_2}{2}, \tau_{z2}}^\dagger a_{p - \frac{k_1 + k_2}{2}, -\tau'_{z1}} \quad (\text{A. 1}) \\
&= \frac{\sqrt{2}}{\sqrt{A}} \sum_{T_3, T_{z3}} \sqrt{(2T_2 + 1)(2T_3 + 1)} W \left(\frac{1}{2} \frac{1}{2} T_2 T_3; T_1 \frac{1}{2} \right) \langle T_3 T_{z3} T_2 - T_{z2} | T_1 - T_{z1} \rangle \\
&\times \frac{\sqrt{2}}{\sqrt{A}} \left\{ \sum_{p, \tau_{z1}, \tau'_{z2}} \left\langle \frac{1}{2} \tau_{z1} \frac{1}{2} \tau'_{z2} | T_3 T_{z3} \right\rangle a_{p + \frac{k_1 + k_2}{2}, \tau_{z1}}^\dagger (-1)^{\frac{1}{2} + \tau'_{z2}} a_{p - \frac{k_1 + k_2}{2}, -\tau'_{z2}} \right. \\
&\quad \left. - (-1)^{T_1 + T_2 + T_3} \sum_{p, \tau_{z2}, \tau'_{z1}} \left\langle \frac{1}{2} \tau_{z2} \frac{1}{2} \tau'_{z1} | T_3 T_{z3} \right\rangle a_{p + \frac{k_1 + k_2}{2}, \tau_{z2}}^\dagger (-1)^{\frac{1}{2} + \tau'_{z1}} a_{p - \frac{k_1 + k_2}{2}, -\tau'_{z1}} \right\} \\
&= \frac{\sqrt{2}}{\sqrt{A}} \sum_{T_3, T_{z3}} \{ (-1)^{T_1 + T_2 + T_3 - 1} \} \sqrt{(2T_1 + 1)(2T_2 + 1)} W \left(\frac{1}{2} \frac{1}{2} T_1 T_2; T_3 \frac{1}{2} \right) \langle T_1 T_{z1} T_2 T_{z2} | T_3 T_{z3} \rangle \rho_{k_1 + k_2}^{T_3, T_{z3}},
\end{aligned}$$

which is just the commutation relation given by (2.11).

From the expressions for $\rho_k^{0,0}$ and $\pi_k^{0,0}$, the commutators among them can be computed as

$$\left. \begin{aligned}
[\pi_k^{0,0}, \rho_{k'}^{0,0}] &= -\frac{i\hbar}{Ak^2} \sum_{p, p'} \sum_{\tau_z} p k \left(a_{p - \frac{k}{2}, \tau_z}^\dagger a_{p + \frac{k}{2}, \tau_z} - a_{p' - \frac{k'}{2}, \tau_z}^\dagger a_{p' + \frac{k'}{2}, \tau_z} \right) = -\frac{i\hbar}{\sqrt{A}} \frac{k'}{k} \rho_{k' - k}^{0,0}, \\
[\pi_k^{0,0}, \pi_{k'}^{0,0}] &= -\frac{i\hbar^2}{Ak^2 k'^2} \sum_{p, p', \tau_z} p k p' k' \left(a_{p - \frac{k}{2}, \tau_z}^\dagger a_{p + \frac{k}{2}, \tau_z} - a_{p' - \frac{k'}{2}, \tau_z}^\dagger a_{p' + \frac{k'}{2}, \tau_z} \right) \\
&= -\frac{i\hbar^2}{Ak^2 k'^2} \sum_{p, \tau_z} \left\{ \left(p - \frac{k}{2} \right) \left(p - \frac{k'}{2} \right) - \left(p + \frac{k}{2} \right) \left(p - \frac{k'}{2} \right) \right\} k k' a_{p - \frac{k + k'}{2}, \tau_z}^\dagger a_{p + \frac{k + k'}{2}, \tau_z} \\
&= -\frac{i\hbar^2}{\sqrt{A} k k'} (k^2 - k'^2) \pi_{k + k'}^{0,0}.
\end{aligned} \right\} (\text{A. 2})$$

B Commutator Among $\pi_k^{0,0}$ and $\Pi_{k'}^{0,0}$ and the Proof of the *Exact* Canonical Commutation Relations and of the Commutativity of $\Pi_k^{0,0}$ for Different k 's

The commutator among $\pi_k^{0,0}$ and $\Pi_{k'}^{0,0}$ is calculated tediously but straightforwardly as

$$\begin{aligned}
[\pi_k^{0,0}, \Pi_{k'}^{0,0}] &= [\pi_k^{0,0}, \pi_{k'}^{0,0}] - \frac{1}{\sqrt{A}k'} \sum_{p \neq k'} p \{ \rho_{p-k'}^{0,0} [\pi_k^{0,0}, \pi_p^{0,0}] + [\pi_k^{0,0}, \rho_{p-k'}^{0,0}] \pi_p^{0,0} \} \\
&+ \frac{1}{\sqrt{A}k'} \frac{1}{\sqrt{A}} \sum_{p \neq k', q \neq p} q \{ \rho_{p-k'}^{0,0} \rho_{q-p}^{0,0} [\pi_k^{0,0}, \rho_q^{0,0}] + (\rho_{p-k'}^{0,0} [\pi_k^{0,0}, \pi_{q-p}^{0,0}] + [\pi_k^{0,0}, \rho_{p-k'}^{0,0}] \rho_{q-p}^{0,0}) \pi_q^{0,0} \} + \dots \\
&= \frac{i\hbar}{\sqrt{A}k} (k+k') \left\{ \pi_{k+k'}^{0,0} - \frac{i\hbar}{\sqrt{A}(k+k')} \left[\sum_{p \neq k+k'} p \rho_{p-k-k'}^{0,0} \pi_p^{0,0} + \frac{1}{\sqrt{A}} \sum_{p \neq k+k', q \neq p} q \rho_{p-k-k'}^{0,0} \rho_{q-p}^{0,0} \pi_q^{0,0} \right. \right. \\
&\quad \left. \left. - \frac{1}{\sqrt{A}} \frac{1}{\sqrt{A}} \sum_{p \neq k+k', q \neq p, r \neq q} r \rho_{p-k-k'}^{0,0} \rho_{q-p}^{0,0} \rho_{r-q}^{0,0} \pi_r^{0,0} \right] \right\} + \dots \\
&= \frac{i\hbar}{\sqrt{A}k} (k+k') \Pi_{k+k'}^{0,0}.
\end{aligned}
\tag{B. 1}$$

Thus, we can get (2.18) and (3.3).

The *exact* canonical commutation relations are proved as follows: First, iterating the discrete integral equation (3.1) and using (2.18), we get

$$\begin{aligned}
[\Pi_k^{0,0}, \rho_{k'}^{0,0}] &= [\pi_k^{0,0}, \rho_{k'}^{0,0}] - \frac{1}{\sqrt{A}k} \sum_{p \neq k} \rho_{p-k}^{0,0} [\pi_p^{0,0}, \rho_{k'}^{0,0}] \\
&+ \frac{1}{\sqrt{A}k} \frac{1}{\sqrt{A}} \sum_{p \neq k, q \neq p} q \rho_{p-k}^{0,0} \rho_{q-p}^{0,0} [\pi_q^{0,0}, \rho_{k'}^{0,0}] \\
&- \frac{1}{\sqrt{A}k} \frac{1}{\sqrt{A}} \frac{1}{\sqrt{A}} \sum_{p \neq k, q \neq p, r \neq q} r \rho_{p-k}^{0,0} \rho_{q-p}^{0,0} \rho_{r-q}^{0,0} [\pi_r^{0,0}, \rho_{k'}^{0,0}] + \dots \\
&= -\frac{i\hbar k'}{\sqrt{A}k} \rho_{k'-k}^{0,0} + \frac{1}{\sqrt{A}k} \sum_{p \neq k} p \rho_{p-k}^{0,0} \frac{i\hbar k'}{\sqrt{A}p} \rho_{k'-p}^{0,0} \\
&- \frac{1}{\sqrt{A}k} \frac{1}{\sqrt{A}} \sum_{p \neq k, q \neq p} q \rho_{p-k}^{0,0} \rho_{q-p}^{0,0} \frac{i\hbar k'}{\sqrt{A}q} \rho_{k'-q}^{0,0} \\
&+ \frac{1}{\sqrt{A}k} \frac{1}{\sqrt{A}} \frac{1}{\sqrt{A}} \sum_{p \neq k, q \neq p, r \neq q} r \rho_{p-k}^{0,0} \rho_{q-p}^{0,0} \rho_{r-q}^{0,0} \frac{i\hbar k'}{\sqrt{A}r} \rho_{k'-r}^{0,0} - \dots \\
&= -\frac{i\hbar k'}{\sqrt{A}k} \rho_{k'-k}^{0,0} + \frac{i\hbar k'}{\sqrt{A}k} \rho_{k'-k}^{0,0} (1 - \delta_{kk'}) \\
&+ \frac{i\hbar k'}{\sqrt{A}k} \frac{1}{\sqrt{A}} \sum_{p \neq k, k' \neq p} \rho_{p-k}^{0,0} \rho_{k'-p}^{0,0} - \frac{i\hbar k'}{\sqrt{A}k} \frac{1}{\sqrt{A}} \sum_{p \neq k, k' \neq p} \rho_{p-k}^{0,0} \rho_{k'-p}^{0,0} - \dots \\
&= -\frac{i\hbar k'}{\sqrt{A}k} \rho_{k'-k}^{0,0} \cdot \delta_{kk'} = -i\hbar \delta_{kk'}, \quad (\rho_0^{0,0} = \sqrt{A}).
\end{aligned}
\tag{B. 2}$$

In each summation of the above, we separate the part with $p = k'$ and that with $p \neq k'$ and use the relation $\rho_0^{0,0} = \sqrt{A}$. All terms which involve higher order powers of $\rho_k^{0,0}$ cancel out except the c -number term in the last line of the RHS of Eq. (B. 2) which arises due to the exclusion of the $p = k'$ term if $k = k'$ in the summation with respect to p . Thus we could reach the *exact* canonical commutation relation. In order to assert that the operators $\Pi_k^{0,0}$'s are *exact* canonically conjugate to $\rho_k^{0,0}$'s, we must give the proof of the commutativity of the $\Pi_k^{0,0}$'s for different k 's. The commutation relation among the $\Pi_k^{0,0}$'s can be calculated tediously but straightforwardly as

$$\begin{aligned}
[\Pi_k^{0,0}, \Pi_{k'}^{0,0}] &= -\frac{i\hbar}{\sqrt{A}kk'}(k^2 - k'^2)\pi_{k+k'}^{0,0} + \frac{i\hbar}{\sqrt{A}kk'}(k^2 - k'^2)\Pi_{k+k'}^{0,0} \\
&\quad - \frac{1}{\sqrt{A}k} \sum_{p \neq k} p \rho_{p-k}^{0,0} [\Pi_p^{0,0}, \pi_{k'}^{0,0}] + \frac{1}{\sqrt{A}k'} \sum_{q \neq k'} q \rho_{q-k'}^{0,0} [\Pi_q^{0,0}, \pi_k^{0,0}] \\
&\quad + \frac{1}{\sqrt{A}k} \frac{1}{\sqrt{A}k'} \sum_{p \neq k} \sum_{q \neq k'} pq \rho_{p-k}^{0,0} \rho_{q-k'}^{0,0} [\Pi_p^{0,0}, \Pi_q^{0,0}] \\
&= -\frac{i\hbar}{\sqrt{A}kk'}(k^2 - k'^2)\pi_{k+k'}^{0,0} + \frac{i\hbar}{\sqrt{A}kk'}(k^2 - k'^2)\Pi_{k+k'}^{0,0} \\
&\quad + \frac{i\hbar}{\sqrt{A}k\sqrt{A}k'} \sum_{p \neq k} p(p+k') \rho_{p-k}^{0,0} \Pi_{p+k'}^{0,0} \\
&\quad - \frac{i\hbar}{\sqrt{A}k\sqrt{A}k'} \sum_{q \neq k'} q(q+k) \rho_{q-k'}^{0,0} \Pi_{q+k}^{0,0} \\
&\quad + \frac{1}{\sqrt{A}k\sqrt{A}k'} \sum_{p \neq k} \sum_{q \neq k'} pq \rho_{p-k}^{0,0} \rho_{q-k'}^{0,0} [\Pi_p^{0,0}, \Pi_q^{0,0}] \\
&= F(k; k') + \frac{1}{\sqrt{A}k\sqrt{A}k'} \sum_{p \neq k} \sum_{q \neq k'} pq \rho_{p-k}^{0,0} \rho_{q-k'}^{0,0} F(q; p) + \dots
\end{aligned} \tag{B. 3}$$

In the above we have introduced the operator-valued function $F(k; k')$ defined below. The definition of the function $F(k; k')$ shows that it self-evidently vanishes exactly owing to the definition of the *exact* canonically conjugate momenta $\Pi_k^{0,0}$, i.e., (3.1),

$$\begin{aligned}
F(k; k') &\equiv -\frac{i\hbar}{\sqrt{A}kk'}(k^2 - k'^2) \left\{ \pi_{k+k'}^{0,0} - \frac{i\hbar}{\sqrt{A}(k+k')} \sum_{p \text{ all}} p \rho_{p-(k+k')}^{0,0} \Pi_p^{0,0} \right\} \\
&= -\frac{i\hbar}{\sqrt{A}kk'}(k^2 - k'^2) \left\{ \pi_{k+k'}^{0,0} - \frac{i\hbar}{\sqrt{A}(k+k')} \sum_{p \neq k+k'} p \rho_{p-(k+k')}^{0,0} \Pi_p^{0,0} - \Pi_{k+k'}^{0,0} \right\} \\
&= -\frac{i\hbar}{\sqrt{A}kk'}(k^2 - k'^2) \{ \Pi_{k+k'}^{0,0} - \Pi_{k+k'}^{0,0} \} = 0.
\end{aligned} \tag{B. 4}$$

Thus, we can give the proof of the commutativity of the $\Pi_k^{0,0}$'s for different k 's.

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